Algebra 2 Notes — Name: key
Section 7.6 – The Natural Base, e

DAY ONE:

In Algebra 1, you learned the compound interest formula. You have probably since forgotten it. ☺ The compound interest formula for the amount A in an account is given below.

rate as a decimal

\# of years

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

resulting amount

\# times compounded per year

principal (starting amt)

\# times compound per year

As n gets very large, the interest is begins to be compounded continuously… all of the time without taking a break. When interest is compounded continuously, the formula above can be simplified using the natural base e. What in the world is the value of the number e?

DISCOVERY: Use your calculator to evaluate the expression $\left(1+\frac{1}{n}\right)^n$ for the following values.

n	2	4	8	32	128	512	1024	5000	10000	50000
$\left(1+\frac{1}{n}\right)^n$	2.25	2.4414	2.5658	2.677	2.7077	2.7156	2.717	2.718	2.7181	2.7183

So, using the idea of a limit, which you learn more about in Pre-Calculus, we can see that the $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. From our discovery, we see that $e \approx 2.718$

The continuously compounded interest formula for the amount A in an account is given below.

rate as a decimal

\# of years

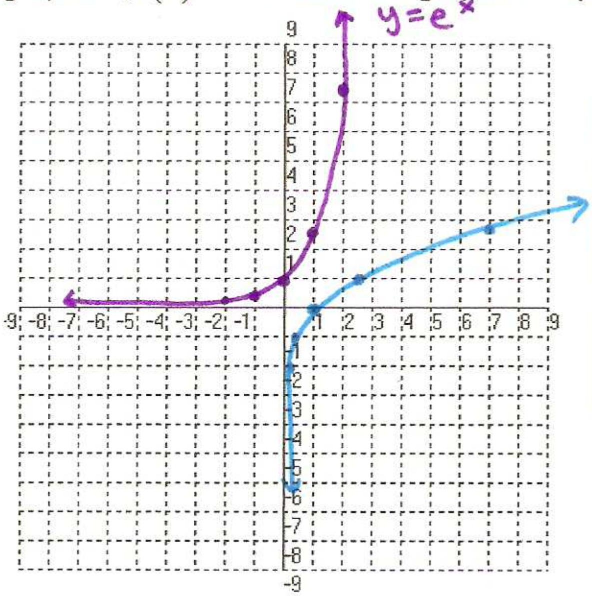
$$A = Pe^{rt}$$

resulting amount

principal (starting amt)

We will use this formula in a later example, but let's start by graphing some functions involving this natural base e.

Exponential functions with e as a base have the SAME properties as the functions you have studied. The graph of $f(x)=e^x$ is like other graphs of exponential functions, such as $f(x)=3^x$.



$y=e^x$

x	y
0	1
1	$e \approx 2.718$
2	$e^2 \approx 7.389$
−1	$e^{-1} \approx 0.368$
−2	$e^{-2} \approx 0.135$

Domain: ℝ

Range: $y>0$

$y=\ln x$

x	y
1	0
2.718	1
7.389	2
0.368	−1
0.135	−2

Domain: $x>0$

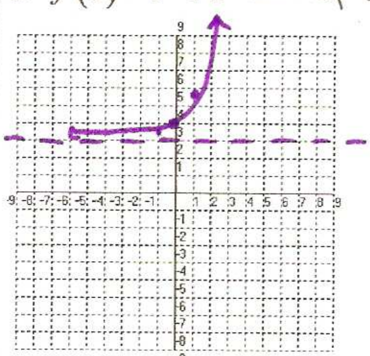
Range: ℝ

A logarithm with a base of e is called a natural logarithm and is abbreviated as "ln x" (rather than $\log_e$). Natural logarithms have the same properties as log base 10 and logarithms with other bases.

The natural logarithmic function $f(x)=\ln x$ is the inverse of the natural exponential function $f(x)=e^x$. All of the properties from Section 7.4 also apply to natural logarithms.

Example 1: Graph. Find domain and range and describe transformations.

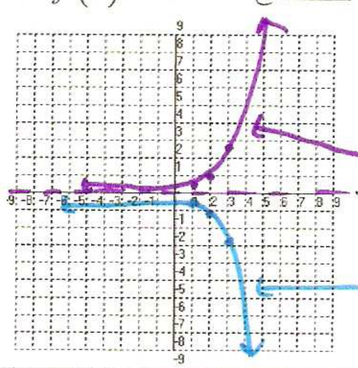
a. $f(x)=e^x+3$ ← up 3



D: ℝ

R: $y>3$

b. $f(x)=-e^{x-2}$ ← reflected across x-axis and shifted 2 units right



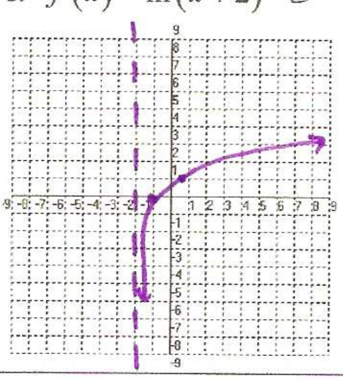
$y=e^{x-2}$

$y=-e^{x-2}$ final graph

D: ℝ

R: $y>0$

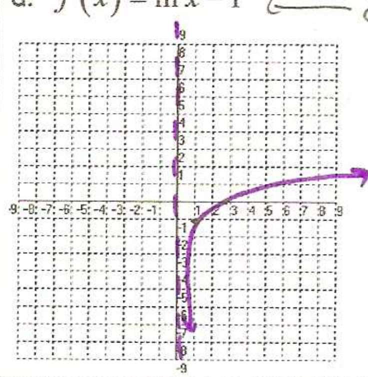
c. $f(x)=\ln(x+2)$ ← 2 to the left



D: $x>-2$

R: ℝ

d. $f(x)=\ln x-1$ ← down 1



D: $x>0$

R: ℝ

Example 2: Simplify.

a. $\ln e^{-2t}$	b. $e^{\ln(t-1)}$	c. $e^{5\ln x}$
$-2t \ln e$ $-2t(1)$ $\boxed{-2t}$	$\boxed{t-1}$	$e^{\ln x^5}$ $\boxed{x^5}$

Now... back to the formula for continuously compounded interest, $A = Pe^{rt}$.

Example 3: Applications.

a. What is the total amount for an investment of \$1000 compounded at 5% for 10 years compounded continuously?	b. What is the total amount for an investment of \$4000 invested at 3.5% for 8 years and compounded continuously?
$A = Pe^{rt}$ $A = 1000e^{0.05(10)}$ $\boxed{A \approx \$1648.72}$	$A = Pe^{rt}$ $A = 4000e^{0.035(8)}$ $A \approx \boxed{\$5292.52}$

DAY TWO:

Example 4: Simplify. Round to nearest hundredth.

a. $\ln 4 - \ln x = 1$	b. $\ln 5 + \ln x^2 = 7$	c. $e^{\ln x^2} = 25$
$\ln \frac{4}{x} = 1$ $e^1 = \frac{4}{x}$ $e = \frac{4}{x}$ $ex = 4$ $x = \frac{4}{e}$ $\boxed{x \approx 1.47}$	$\ln 5x^2 = 7$ $e^7 = 5x^2$ $\sqrt{\frac{e^7}{5}} = \sqrt{x^2}$ $x = \pm\sqrt{\frac{e^7}{5}}$ $\boxed{x \approx \pm 14.81}$	$\sqrt{x^2} = \sqrt{25}$ $\boxed{x = \pm 5}$

The half-life of a substance is the time it takes for half of the substance to breakdown or convert to another substance during the process of decay. Natural decay is modeled by the function below.

$$N(t) = N_0 e^{-kt}$$

amount remaining → $N(t)$; initial amount @ $t=0$ → N_0; decay constant → k; time → t

Easier formula:

time/amt. of time for half life → x

$$y = a\left(\tfrac{1}{2}\right)^x \quad \text{OR} \quad y = a(0.5)^x$$

amt remaining → y; initial amount → a

Example 5: Applications.

a. A paleontologist uncovers a fossil of a saber-toothed cat in Bexar County. He analyzes the fossil and finds that the specimen contains 15% of its original carbon-14. Carbon-14 has a half-life of 5730 years. Use carbon-14 dating to determine the age of the fossil.

$$15 = 100\left(\tfrac{1}{2}\right)^{x/5730}$$

$$.15 = \left(\tfrac{1}{2}\right)^{x/5730}$$

$$\log_{0.5} 0.15 = \frac{x}{5730}$$

$$x = 5730 \log_{0.5} 0.15$$

$$x = 5730\left(\frac{\log 0.15}{\log 0.5}\right)$$

$$\boxed{x \approx 15{,}683 \text{ years old}}$$

b. Determine how long it will take for a 650 mg of a sample of chromium-51, which has a half-life of about 28 days, to decay to 200 mg.

$$200 = 650\left(\tfrac{1}{2}\right)^{x/28}$$

$$\frac{4}{13} = \left(\frac{1}{2}\right)^{x/28}$$

$$\log_{0.5}\frac{4}{13} = \frac{x}{28}$$

$$x = 28\log_{0.5}\frac{4}{13}$$

$$x = 28\left(\frac{\log \frac{4}{13}}{\log 0.5}\right)$$

$$\boxed{x \approx 48 \text{ days}}$$